**Homework 4**

**Instructions:** Do as many of the problems as you like, but make sure to complete at least **three**. Then I will create a solution from your work.

1. Find two other famous number theory conjectures. Briefly describe what the conjecture says and add any information about the current status of the work on this conjecture.

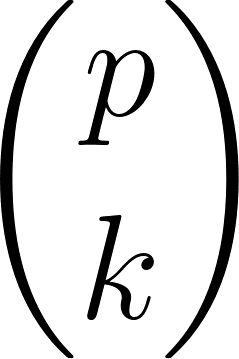
* Two of [Landau's Problems](https://en.wikipedia.org/wiki/Landau%27s_problems) (quotes taken from the wikipedia page):
* Does there always exists a prime number between consecutive square numbers?
  + "the conjecture holds to . A counterexample near would require a prime gap fifty million times the size of the average gap."
  + "A result due to Ingham shows that there is a prime between and for every large enough ."
* Are there infinitely many prime numbers of the form ?
  + "Henryk Iwaniec showed that there are infinitely many numbers of the form with at most two prime factors"
  + "Merikoski ... showed that there are infinitely many numbers of the form with greatest prime factor at least . Replacing the exponent with 2 would yield Landau's conjecture."

1. Find the prime factorization of 185803 by hand. (Briefly explain your method for finding the prime factorization.)

* The number is clearly not even. We can quickly check that 3 does not divide it because 1+8+5+8+0+3=25 which 3 does not divide. It is clearly not divisible by 5 either.
* From here I checked division of 185803 by 7, 11, 13, 17, 19, 23, and 29 (the next primes) using my calculator and checking if the result was an integer. The first to work was 29.
* 185803 / 29 = 6407 so from here I checked division of 6407 by the next primes (also checking if 29 worked again): 29, 31, 37, 41, 43. The first of these to work was 43.
* 6407 / 43 = 149. 149 is larger than the square of the primes that I am up to (47) and no smaller primes can divide 149 or else they would have divided 6407 or 185803. So 149 must be prime.
* Thus the prime factorization of 185803 is 29\*43\*149.

1. Show that there are infinitely many prime numbers of the form (The same idea applies for prime numbers of the form )

* Five is a prime number of the form so we know there exists at least one.
* Assume there are finitely many primes of the form .
* Consider .
* We know for each so no divides . Three also does not divide so each prime factor of must have a remainder of 1 modulo 3.
* But the product of numbers of the form will have a remainder of 1.
* We know that so we have reached a contradiction.

1. Show that the binomial coefficient [](https://www.codecogs.com/eqnedit.php?latex=%5Cbinom%7Bp%7D%7Bk%7D#0) is divisible by *p* if *p* is prime and
2. Let be the *n*th prime number.
3. Show that

* For each where , we know and so does not divide it.
* Thus, either is a prime larger than or else each of its prime factors is larger than .
* Thus, either is prime or else there exists a prime between and and so

1. Use part a. to show that (Hint: Use induction.)
2. Updated: Use part a. to show that . (Hint: Use induction.)

* I will inductively prove that which implies .
* Basis: because .
* Strong Induction: Assume that for all for some . Show that .
  + From part a we know that . By assumption we know that for each .
  + Combining these facts yields .
  + Simplifying this yields .
  + The division by 2 reduces the expression's value by at least 1 since for . Thus, removing the division by 2 as well as the plus 1 will either not change the expression's value at all or it will increase it. Mathematically, .
  + Thus, .
* We have thus inductively proven that for . This then implies that .

1. Write a code to implement the Eratosthenes Sieve to create the list of all primes up to *n* for given *n*.

* def sieve(n):
  + all\_ints = [i for i in range(2, n+1)]
  + primes = []
  + for i in range(n-1):
    - # if the integer at index i has not been ``crossed out'' already
    - if all\_ints[i] != 0:
      * # then that integer must be prime
      * prime = all\_ints[i]
      * primes.append(prime)
      * j = i
      * # ``cross'' out its multiples
      * while j + prime < n - 1:
        + all\_ints[j + prime] = 0
        + j += prime
  + return primes